MOTION OF A VISCOUS LIQUID BETWEEN TWO ROTATING SPHERES

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An analysis is presented of the steady-state asymmetric motion of an incompressible viscous liquid between two concentric spheres rotating with constant angular velocities about various axes passing through their common center. The reaction force of the liquid on the inner sphere is determined; this force reduces to a resistive torque.

Let the radii of the spheres be r_1 and r_2 ($r_1 < r_2$), their angular velocities ω_1 and ω_2, and the angle between their axes β. In a spherical coordinate system r, φ, θ, with the axis of rotation of the inner sphere along the line $\theta = 0$ and that of the outer sphere in the plane $\varphi = 0$, the boundary conditions are ($v_i = \omega_i r_i$, i = 1, 2)

$$\begin{aligned} &v_\varphi = v_1 \sin\theta,\ v_r = 0,\ v_\theta = 0 && \text{for } r = r_1 \\ &v_\varphi = v_2 (\cos\beta \sin\theta - \sin\beta \cos\theta \cos\varphi),\ v_r = 0,\ v_\theta = -v_2 \sin\beta \sin\varphi && \text{for } r = r_2 \end{aligned} \tag{1}$$

We look for the solution of the Navier–Stokes and continuity equations [1] for (1) in dimensionless coordinates in the form of power series in the Reynolds number R, which converge for small R [2, 3], and which have coefficients that can be found by the method of separation of variables and that can be expressed in terms of elementary functions as in [3, 4]. Thus, for example,

$$v_r = v_1 \sum_{k=1}^{\infty} R^{2k} \sum_{n=1}^{2k} \sum_{i=1}^{k} v_{2k,n,2i}(\xi) P_{2i}^{n}(\tau) \sin n\varphi + {}$$
$$+ v_1 \sum_{k=1}^{\infty} R^{2k-1} \sum_{n=0}^{2k} \sum_{i=0}^{k} v_{2k-1,n,2i}(\xi) P_{2i}^{n}(\tau) \cos n\varphi$$

Here $R = r_1 v_1/\nu$, $\xi = r/r_1$, $\tau = \cos\theta$, $P_m^n(\tau)$ are associated Legendre functions. PU

Knowing these solutions, we can show that the principal force vector for the interaction between the inner sphere and the liquid is identically equal to zero and that the projections of the resistive torque M for this sphere on the axes of a Cartesian system with the z axis along $\theta = 0$ and the x axis in the plane $\varphi = 0$ have the following values to an accuracy of terms in R^2:

$$\begin{gathered} M_x = 8\pi\mu v_1 r_1^2 b \delta_0 \sin\beta \, [a^{-1} + R^2 \delta_1 (\delta_7 - b\delta_5 \cos\beta + b^2 \delta_6)], \quad M_y = 0 \\ M_z = -8\pi\mu v_1 r_1^2 \delta_0 \{1 - a^{-1} b \cos\beta + R^2 \delta_1 [\delta_2 + b\delta_3 \cos\beta + b^2 (\delta_4 - \delta_5 \sin^2\beta) - b^3 \delta_6 \cos\beta]\} \end{gathered} \tag{2}$$

$$a = \frac{r_2}{r_1}, \quad b = \frac{v_2}{v_1}, \quad \delta_0 = \frac{a^3}{\delta}, \quad \delta_1 = \frac{a(a-1)^7}{300 \Delta \delta^3}, \quad \delta = a^3 - 1$$

$$\begin{aligned} &\Delta = 4a^6 + 16a^5 + 40a^4 + 55a^3 + 40a^2 + 16a + 4, \quad \delta_2 = a^7 + 11a^6 + \\ &+ 66a^5 + 146a^4 + 136a^3 + 45a^2, \quad \delta_3 = 37a^6 + 182a^5 + 237a^4 + 47a^3 - \\ &- 98a^2 - 45a, \ \delta_4 = 45a^6 + 98a^5 - 47a^4 - 237a^3 - 182a^2 - 37a, \ \delta_5 = \\ &= 0.375 (30a^6 + 2a^5 - 353a^4 - 663a^3 - 443a^2 - 88a), \quad \delta_6 = 45a^5 + \\ &+ 136a^4 + 146a^3 + 66a^2 + 11a + 1, \ \delta_7 = 0.125 (-32a^6 - 127a^5 + 93a^4 + 683a^3 + 778a^2 + 270a) \end{aligned}$$

The limit of (2) for $\omega_2 = 0$ and $a \to \infty$ leads to the well-known results for a sphere rotating in an unbounded viscous liquid [3].

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